1. *If $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous functions then $h = \min\{f, g\}$ is also a continuous function.*

   Solution: For any $a, b \in \mathbb{R}$ we have $\min\{a, b\} = \frac{1}{2}\big[(a + b) - |a - b|\big]$. So using this we get

   $$h(x) = \frac{1}{2}\big[\big(f(x) + g(x)\big) - \big|f(x) - g(x)\big|\big].$$

   Now continuity of $f + g$ and $|f - g|$ will give the continuity of h. □

2. *Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots\cdots$ where $a_0, a_1, a_2 \cdots$ are real numbers. Let $\{\limsup |a_n|^{\frac{1}{n}}\}^{-1} = R_0$*
   (i) Find R_0 when $a_n = \frac{1}{n!}$.
   (ii) Give example of $a_0, a_1, a_2 \cdots$ such that $R_0 = 1$, for $|x| = R_0$, the series $\sum_{n=0}^{\infty} a_nx^n$ is divrgent for $x = 1$ and convergent for $x = -1$.

   Solution: (i) We have the following power series

   $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

   with $a_n = \frac{1}{n!}$, converges for all $x \in \mathbb{R}$. So we get $R_0 = \infty$.
   Solution: (ii) Let $a_0 = 0$ and $a_n = -\frac{1}{n}$, $n \geq 1$ then we have

   $$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

   which converges for $-1 \leq x < 1$ and diverges for $x = 1$. □

3. *Let $I_1 \supset I_2 \supset \cdots$ be a sequence of closed intervals with length $I_r \longrightarrow 0$ as $r \to \infty$. If $x_j \in I_j$ then show that the sequence $\{x_1, x_2, x_3, \dots\}$ is a cauchy sequence.*

   Solution: We have $x_{n+k} \in I_{n+k} \subset I_n$ for any $n \geq 1$ and $k = 1, 2, \cdots$. Thus we get

   $$|x_{n+k} - x_n| \leq |I_n| \;\; as\ x_n, x_{n+k} \in I_n,\ here\ |I_n|\ is\ length\ of\ I_n.$$

   So from above we get $|x_{n+k} - x_n| \to 0$ *as* $n \to \infty$, for $k = 1, 2, \cdots$. This prove that $\{x_n\}_{n\geq 1}$ is a cauchy sequence. □

4. *Let $x_1, x_2, \cdots$ be bounded sequence of reals with $x_j \geq 0$. If every subsequence of x_n has a [further] subsequence converging to 0 show that $x_n \to 0$.*

   Solution: Let assume by that x_n does not convergent to 0. Then there is a $\epsilon > 0$ such that for any $M \in \mathbb{N}$ there is a x_m with $|x_m| \geq \epsilon$ and $m > M$.
   Now chose the subsequence $\{x_{n_k}\}_k$ with $|x_{n_k}| \geq \epsilon$ with $n_k > N_k$ and $N_{K+1} > N_k$. Then this susequence $\{x_{n_k}\}_k$ does not admit any further subsequence which converges to 0. Which give the contradiction. Hence we have $x_n \to 0$. □

5. *(a) Let* $a_n > 0$, $\sum_n a_n^2 < \infty$, $\partial > \frac{1}{2}$. *Then show that* $\sum_1^\infty \frac{a_n}{n^\partial}$ *exists.*
(b) $\mathbb{B} > \frac{1}{2}$. *Show that* $\sum_n b_n < \infty$ *where* $b_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^{\mathbb{B}}}$.
(c) If $|x| < 1$ *show that* $x^n \to 0$ *as* $n \to \infty$.

Solution (a) Let $S_N = \sum_{n=1}^{N} \frac{a_n}{n^\partial}$ then by cauchy schwarz-inequality we have

$$S_N \leq \sqrt{\sum_{n=1}^{N} a_n^2}\sqrt{\sum_{n=1}^{N} c_n^2} \leq \sqrt{\sum_{n=1}^{\infty} a_n^2}\sqrt{\sum_{n=1}^{\infty} c_n^2} < \infty \ \ \forall\ N,$$

here $c_n = \frac{1}{n^\partial}$. Since $\partial > \frac{1}{2}$ we have $\sum_{n=1}^{\infty} c_n^2 < \infty$. The above inequality give the existence of $\sum_1^\infty \frac{a_n}{n^\partial}$.

Solution (b) Consider $c_n = \frac{1}{n^{\mathbb{B}+\frac{1}{2}}}$ then $\sum_{n=1}^{\infty} c_n < \infty$ as $\mathbb{B}+\frac{1}{2} > 1$. It is easy to see that $\lim_{n\to\infty} \frac{b_n}{c_n} = \frac{1}{2}$. Now comparison test will give the converges of $\sum_n b_n$.

Solution (c) Let $|c_0| < 1$ and $0 < \epsilon < 1$ (it is enough) then we have to find a $N \in \mathbb{N}$ such that $|c_0|^n < \epsilon \ \ \forall\ n > M$. If $|c_0|^n < \epsilon$ then we have $n > \frac{log\epsilon}{logc_0}$. So if we take $M = [\frac{log\epsilon}{logc_0}] + 1$ then we are done. □

6. *(a) Show that* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is given by* $f(x) = x^2$ *is not uniformly continuous.*
(b) Let f *be as in (a). Show that if* $x_1, x_2, \cdots$ *is a cauchy sequence then* $f(x_1), f(x_2), \cdots$ *is a cauchy sequence.*
(c) Give an example of uniformly continuous functions g_1, g_2 *such that the product* g_1g_2 *is not uniformly continuous and prove your claim.*
(d) Let $g : J \longrightarrow \mathbb{R}$ *be uniformly continuous. Show that if* $x_1, x_2, \cdots$ *is a cauchy sequence in* J *then* $g(x_1), g(x_2), \cdots$ *is a cauchy sequence.*
(e) Let $h : (0,1] \longrightarrow \mathbb{R}$ *be uniformly continuous. If* $y_n \in (0,1]$ *and* $y_n \to 0$ *then* $\{h(y_n)\}$ *is convergent and the limit is independent of the sequence* $y_1, y_2, \cdots$ *(converging to 0)*
(f) Let $k : J \longrightarrow \mathbb{R}$ *be a continuous, differentiable function and the derivative be bounded and continuous. Show that* k *is uniformly continuous. Here* J *is a bounded or unbounded interval.*

Solution (a) Let $0 < \epsilon < 1$. For any given $\delta > 0$ set $x_1 = \frac{1}{\delta}$ and $x_2 = x_1 + \frac{\delta}{2}$ then $|x_1 - x_2| = \frac{\delta}{2} < \delta$. Now

$$|f(x_1) - f(x_2)| = 1 + \frac{\delta^2}{4} > 1 > \epsilon.$$

The above give that f is not uniformly continuous.

Solution (b) Since $\{x_n\}_n$ is a cauchy sequence so it is bounded, let $|x_n| \leq M \ \ \forall\ n$. Now

$$|f(x_n) - f(x_m)| = |(x_n + x_m)(x_n - x_m)| \leq 2M|x_n - x_m|.$$

So $f(x_1), f(x_2), \cdots$ is a cauchy sequence.

Solution (c) Consider $g_1(x) = g_2(x) = x$ is uniformly continuous (take $\epsilon = \delta$). But from (a) we have $g_1g_2(x) = x^2$ is not uniformly continuous.

Solution (d) Since $\{x_n\}_n$ is a cauchy sequence, we have for each $\delta > 0$ there is $M \in \mathbb{N}$ (M depends on δ) such that $|x_n - x_m| < \delta, \;\; \forall\, n, m > M$. Now uniform continuity of g will give

$$|g(x_n) - g(x_m)| < \epsilon \;\; \forall\, n, m > M.$$

So $\{g(x_n)\}_n$ is a cauchy sequence.

Solution (e) Since $y_n \to 0$ so $\{y_n\}$ is a cauchy sequence. Now from (d) we have $\{h(y_n)\}$ is also cauchy in $\mathbb{R}$ so it is convergent.
Let $\{x_n\} \in (0, 1]$ be a another sequence such that $x_n \to 0$. Set $x = \lim_{n\to\infty} h(x_n)$ and $y = \lim_{n\to\infty} h(y_n)$. Now for any $\delta > 0$ we can find $M \in \mathbb{N}$ such that $|x_n| < \frac{\delta}{2}$ *and* $|y_n| < \frac{\delta}{2}\; \forall\, n \geq M$. Now uniform continuity of h give that, for any $\epsilon > 0$ we have

$$|h(x_n) - h(y_n)| < \epsilon \;\; as \;\; |x_n - y_n| \leq |x_n| + |y_n| < \delta.$$

Now above will give $|x - y| < \epsilon$ for any $\epsilon > 0$ which imply $x = y$. So the limit of $\{h(y_n)\}$ is independent of $\{y_n\}$.

Solution (f) Using mean value theorem we have

$$|f(x) - f(y)| \leq f'(c)|x - y| \leq M|x - y|, \quad c \in [x, y] \subset J, \quad M = \sup_{u\in J} |f'(u)|.$$

Here we take f is not a constant function. So given $\epsilon > 0$ if we take $\delta = \frac{\epsilon}{M}$ we get

$$|f(x) - f(y)| < \epsilon \;\; whenever \;\; |x - y| < \delta.$$

□

7. *If $a_1, a_2, \cdots$ be a sequence of real with $\sum_n |a_n| < \infty$ then $\sum_n a_n$ exists.*

   Solution We have $0 \leq a_n + |a_n| \leq 2|a_n|$ so $\sum_n (a_n + |a_n|) < \infty$. Now

   $$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (a_n + |a_n|) - \sum_{n=1}^{N} |a_n|$$

   Now $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exists as $\lim_{N\to\infty} \sum_{n=1}^{N} (a_n + |a_n|)$ and $\lim_{N\to\infty} \sum_{n=1}^{N} |a_n|$ both exist. □

8. *Let $a_1,\; a_2,\; \cdots$ be a sequence of reals. $s_n = a_1 + a_2 + \cdots + a_n$. Assume that the sequence s_{3n} is convergent. Then $\sum_n a_n$ exists $\Leftrightarrow a_r \to 0$ as $r \to \infty$.*

   Solution Let $\sum_n a_n$ exist then $s_{r+1} - s_r = a_r \to 0$ as $r \to \infty$ as $\{s_n\}_n$ is a cauchy sequence.
   Now assume that s_{3n} is convergent and $a_r \to 0$ as $r \to \infty$ then $s_{3n+k} - s_{3n} = \sum_{i=1}^{k} a_{3n+i} \to 0$ as $n \to \infty$ for $k = 1,\; 2$. So we have

   $$\lim_{n\to\infty} s_{3n} = \lim_{n\to\infty} s_{3n+1} = \lim_{n\to\infty} s_{3n+2}.$$

   Observe that $\mathbb{N} = \{3n\}_{n=0}^{\infty} \cup \{3n+1\}_{n=0}^{\infty} \cup \{3n+2\}_{n=0}^{\infty}$, this together with above will give that $\lim_{n\to\infty} s_n$ exists i.e $\sum_n a_n < \infty$.

9. *Let* $a_n > 0$ *and* $\sum_n a_n$ *is divergent. Let* $b_n = \frac{a_n}{1+a_n}$*. Show that* $\sum_n b_n$ *is divergent.*

Solution We have $\overline{\lim_{n\to\infty}} \dfrac{a_n}{b_n} = 1 + \overline{\lim_{n\to\infty}}\, a_n = l > 0$. Now for each $\epsilon > 0$ there is $M \in \mathbb{N}$ such that $\frac{a_n}{b_n} < l + \epsilon \ \ \forall\ n > M$. The following inequality gives the divergent of $\sum_n b_n$ as $\sum_n a_n$ is divergent.

$$\sum_{n=M+1}^{\infty} b_n > \frac{1}{l+\epsilon} \sum_{n=M+1}^{\infty} a_n.$$

□

10. *Let* $a_n,\ b_n > 0$ *and* $a_n \to a$ *with* $a \neq 0$*. Show that* $\limsup(a_n b_n) = a \limsup b_n$*.*

Solution Since $a_n \to a$, so we can have $\epsilon > 0$ and $M \in \mathbb{N}$ such that $0 < a - \epsilon < a_n < a + \epsilon \ \forall\ n > M$. So we get $(a-\epsilon)b_n < a_n b_n < (a+\epsilon)b_n \ \ \forall\ n > M$, which will give

$$(a-\epsilon) \lim_{n\to\infty} \sup_{k\geq n} b_n \leq \lim_{n\to\infty} \sup_{k\geq n} a_n b_n \leq (a+\epsilon) \lim_{n\to\infty} \sup_{k\geq n} b_n$$

Let $0 \leq limsup\ b_n = b < \infty$ then above will give $limsup\ (a_n b_n) = ab$. If $limsup\ b_n = \infty$ then L.H.S of the above will give $limsup\ (a_n b_n) = \infty$. □

11. *Let* $x_1,\ x_2,\ \cdots$ be a bounded sequence and $\mathbb{B} = limsup\ x_n$. If $\epsilon > 0$, show that $(\mathbb{B}+\epsilon, \infty)$ can have only finitely many of the $x_1,\ x_2,\ \cdots$.

Solution We have $\mathbb{B} = \inf_n \sup_{k\geq n} x_n = \inf_n A_n$. Here $A_n = \sup_{k\geq n} x_n$ is a non-increasing sequence.Using definition of infimum we get for any $\epsilon > 0$ there is a n_0 such that $A_{n_0} < \mathbb{B} + \epsilon$. Thats give $\sup_{k\geq n_0} x_n < \mathbb{B} + \epsilon$, so $(\mathbb{B}+\epsilon, \infty)$ can contain atmost $\{x_n\}_{n=1}^{n_0}$. □